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CONJUGATE LOCI OF TOTALLY GEODESIC SUBMANIFOLDS OF SYMMETRIC SPACES

J. M. BURNS

ABSTRACT. The conjugate and cut loci of fixed point sets of involutions which fix the origin of a compact symmetric space are studied. The first conjugate locus is described in terms of roots and weights of certain representations. When the first conjugate locus and the cut locus agree, we study Morse functions which give a simple decomposition of the symmetric space. We describe for some examples the topological implications of our results.

INTRODUCTION

This paper will be concerned with the study of conjugate and cut loci of fixed point sets of involutions which fix the origin of a compact symmetric space. We will first describe the conjugate loci of these submanifolds and then give a description of their first conjugate loci in terms of roots, and weights of certain representations. We will show that in some cases the first conjugate locus and the cut locus agree. This being the case, we obtain a Morse function which gives a particularly simple decomposition of the symmetric space. We note that by taking our involution to be the symmetry at the origin we can obtain results on the conjugate locus of a point, a subject which has been studied in detail by Wong [W, 1], Sakai [S, 1], Takeuchi [T, 1], Bott and Samelson [B-S, 1] and Crittenden [Cr, 1] among others. We outline by use of some examples how such a decomposition can be used to calculate homology. The results obtained here comprise part of the author's Ph.D. thesis and he would like to thank Professor T. Nagano for his encouragement and expert guidance. The author would also like to thank the referee for the many helpful suggestions.

PRELIMINARIES

Definition. A Riemannian manifold M is called a symmetric space, if for each point q of M, there exists an involutive isometry s_q of M such that q is an isolated fixed point of s_q. We call s_q the symmetry at q. If G is the closure in the compact open topology of the group of symmetries generated by $\{s_q | q \in M\}$, then it is known that G is transitive on M, (provided M is connected) and hence the isotropy subgroup K, say at 0, is compact and $M = G/K$. We will assume throughout that G is semisimple and that M (and therefore G) is compact. We will denote the Lie algebra of G by $\mathbf{g}$ and the

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involution $\mathrm{ad}(s_0)(g) = s_0 g s_0$ by $\mathrm{ad}(s_0)$. We will use the same notation for the induced involution on $\mathbf{g}$. Since G is semisimple, the Killing form $B_{\mathbf{g}}$ is a negative definite bilinear form on $\mathbf{g}$ invariant under $\mathrm{ad}\, G$.

The Cartan decomposition of $\mathbf{g}$ with respect to $\mathrm{ad}(s_0)$ is given by $\mathbf{g} = \mathbf{k} + \mathbf{m}$ where $\mathbf{k}$ and $\mathbf{m}$ denote the eigenspaces of plus and minus one respectively. Since $\mathbf{k}$ is the Lie algebra of K, we identify $\mathbf{m}$ with the tangent space to M at 0. The following inclusions are well known.

$$[\mathbf{m}, \mathbf{m}] \subset \mathbf{k}, \qquad [\mathbf{k}, \mathbf{m}] \subset \mathbf{m}, \qquad [\mathbf{k}, \mathbf{k}] \subset \mathbf{k}.$$

We will denote by $\langle\ ,\ \rangle$ the unique Riemannian metric on M, which is invariant under G and coincides on the tangent space $T_0 M$ to M at 0 with $-B_{\mathbf{g}}$. Recall that on a symmetric space M the exponential map $\mathrm{Exp}\colon T_0 M \to M$ is given by $\mathrm{Exp}\, X = (\exp X)(0)$, where $\exp$ is the exponential map of G, sometimes just written as e^X, for $X \in \mathbf{m}$. We will denote by $\mathbf{h}$ a maximal abelian subalgebra of m and by A its image under $\mathrm{Exp}\colon T_0 M \to M$, that is A is a maximal torus through 0 in M. The dimension of such a torus is then by definition the rank of M denoted by $r(M)$. Using the fact that $\{\mathrm{ad}(H)^2 | H \in \mathbf{h}\}$ is a commutative system of semisimple operators stabilizing $\mathbf{m}$ and $\mathbf{k}$ we get the following well-known result (see [H1, Chapter 7]).

Theorem A. (1) *We have the following root space decompositions*

$$\mathbf{m} = \mathbf{h} + \sum_{\alpha \in R(M)} \mathbf{m}_\alpha, \qquad \mathbf{k} = \mathbf{k}_0 + \sum_{\alpha \in R(M)} \mathbf{k}_\alpha.$$

(2) *For each* $\alpha \in R(M)$ *we can choose bases* $\{X_\alpha\}$ *and* $\{Y_\alpha\}$ *for* $\mathbf{m}_\alpha$ *and* $\mathbf{k}_\alpha$ *respectively such that*

(a) $\quad [H, X_\alpha] = \alpha(H) Y_\alpha$ *and* $[H, Y_\alpha] = -\alpha(H) X_\alpha$ *for all* $H \in \mathbf{h}$,

(b) $$\mathrm{ad}(\exp H) X_\alpha = \cos\alpha(H) X_\alpha + \sin\alpha(H) Y_\alpha$$

and

$$\mathrm{ad}(\exp H) Y_\alpha = \cos\alpha(H) Y_\alpha - \sin\alpha(H) X_\alpha \quad \textit{for all } H \in \mathbf{h}.$$

Definition. The linear forms $\alpha\colon \mathbf{h} \to \mathbb{R}$ are called the roots of M with respect to $\mathbf{h}$ and $\mu(\alpha) = \dim \mathbf{m}_\alpha = \dim \mathbf{k}_\alpha$ is called the multiplicity of α. We can order the roots in a standard manner and this enables us to define positive roots and simple roots. We will denote the simple roots by $\alpha_1, \dots, \alpha_r$ where $r = r(M)$ and we will use the symbol Σ to denote the set of simple roots.

Definition. A Weyl chamber is a connected component of $\mathbf{h} - \bigcup \ker\alpha$ where the union taken is over all roots of M.

The normalizer and centralizer of $\mathbf{h}$ in K will be denoted as follows. $N(\mathbf{h}, K) = \{k \in K | \mathrm{ad}(k)\mathbf{h} = \mathbf{h}\}$, $C(\mathbf{h}, K) = \{k \in K | \mathrm{ad}(k)_{|\mathbf{h}} = 1_{\mathbf{h}}\}$. The Weyl group will be denoted by W and is defined to be the quotient $N(\mathbf{h}, K)/C(\mathbf{h}, K)$. The following facts can be found in [H1, Chapter 7].

The diagram $\mathscr{D}$ is defined to be the following union of hyperplanes $\mathscr{D} = \bigcup \alpha^{-1}(\pi\mathbb{Z})$ where the union is taken over all α. A connected component of $\mathbf{h} - \mathscr{D}$ is called a fundamental cell of $\mathbf{h}$. The closure of the unique fundamental cell containing 0 and contained in the closed positive Weyl chamber will be called the fundamental Weyl simplex and will be denoted by S, and S therefore has the following description when M is irreducible.

$S = \{H \in \mathbf{h} | 0 \leq \alpha(H) \leq \pi$ for all $\alpha \in \Sigma \cup \widetilde{\alpha}\}$ where $\widetilde{\alpha}$ denotes the highest root. The following is known [T, 1].

Theorem B. $M = K\,\mathrm{Exp}\, S = \mathrm{Exp}\,\mathrm{ad}(K)S$.

Finally we point out some of the geometric meanings of the roots.

Theorem C. *If R denotes the Riemannian curvature tensor of M, then*

$$R(X, Y)Z(0) = -[[X, Y], Z]$$

for $X, Y, Z \in \mathbf{m}$ and $\nabla R = 0$.

Corollary D. *If $H \in h$ and $X \in \mathbf{m}_\alpha$ then $R(X, H)H = \alpha(H)^2 X$.*

1

In this section we will give a description of the conjugate locus of certain totally geodesic submanifolds of M. We begin by making the necessary definitions and recalling some basic facts. A vector field V along a geodesic γ is called a Jacobi field along γ, if it is the variation vector field of a variation of γ through geodesics. This is equivalent to V satisfying the Jacobi equation $\nabla_{\gamma'}\nabla_{\gamma'}V + R(V, \gamma')\gamma' = 0$ where $\nabla_{\gamma'}$ denote covariant differentiation in the direction of γ. We say that $\gamma(s_c)$ is conjugate to $0 = \gamma(0)$ if there exists a nontrivial Jacobi field V along γ vanishing at 0 and at $\gamma(s_c)$, and we say that the tangent vector $s_c\gamma'(0)$ at 0 is a tangent conjugate point of 0. We will assume throughout that geodesics are parametrized by arclength. The set of all tangent conjugate points of 0 is called the tangent conjugate locus of 0 and its image under the exponential map $\mathrm{Exp}\colon T_0M \to M$ is called the conjugate locus of 0 in M, denoted by $JL(\{0\}, M)$. If s_f is the first parameter value for which $\gamma(s)$ is conjugate to 0 along γ, then we say that $\gamma(s_f)$ is a first conjugate point of 0 along γ and that $s_f\gamma'(0)$ is a first tangent conjugate point of 0 along γ. The set of all first tangent conjugate points of 0 is called the first tangent conjugate locus of 0 and it's image under the exponential map $\mathrm{Exp}\colon T_0M \to M$ is called the first conjugate locus of 0 in M, denoted by $FL(\{0\}, M)$. Let γ be a geodesic normal to an embedded submanifold N of M, then following Ambrose [A, 1], we generalize the notion of a conjugate point of 0 along γ to that of a conjugate point of N along γ. In order to do so we consider variation vector fields of variations of γ through geodesics initially normal to N. This is equivalent to saying that the variation vector field V is initially tangent to N and satisfies the initial condition that $\nabla_{\gamma'}V(\gamma(0)) - S_{\gamma'(0)}V(\gamma(0))$ is normal to N, where S is the second fundamental form of N. From now on we will usually denote the covariant derivative of V in the direction of γ by V'.

Definition. We say that $x = \gamma(s_c)$ is a conjugate point of N along $\gamma\colon (\mathbb{R}, 0) \to (M, N)$ if there exists a nontrivial Jacobi field V along γ, satisfying the above initial conditions and vanishing at x. The conjugate locus of N in M will be denoted by $JL(N, M)$, and is defined as follows,

$$JL(N, M) = \{x \in M | x \text{ is conjugate to } N \text{ along some geodesic } \gamma \text{ normal to } N\}.$$

We say that $x = \gamma(s_f)$ is a first conjugate point of N along γ if there exists a nontrivial Jacobi field satisfying the above initial conditions vanishing at x, but no such Jacobi field along γ vanishes at $\gamma(s)$ for $s \in (0, s_f)$. The first conjugate locus of N in M will be denoted by $FL(N, M)$ and is defined as follows,

$$FL(N, M) = \{x \in M | x \text{ is a first conjugate point of } N \text{ along some geodesic}\}.$$

Let t be an involutive isometry of M such that $t(0) = 0$, and let M^t be the component of the fixed point set of t through 0, that is $M^t = F(t, M)(0)$, then by a well-known theorem of Kobayashi [K, 1], M^t is a totally geodesic submanifold of M and is therefore a symmetric space. Let G^t be the identity component of the fixed point set of $\mathrm{ad}(t)$ in G, that is $G^t = F(\mathrm{ad}(t), G)_{(1)}$, then $F(\mathrm{ad}(t), G)$ contains all the symmetries s_x for $x \in M^t$, since if x in M^t, we have that $s_x(x) = x$ and $t(x) = x$, also on T_xM $ds_x = -1$ therefore $ds_x \circ dt = dt \circ ds_x$ which implies that $d(s_x \circ t) = d(t \circ s_x)$. We now have that $s_x \circ t$ and $t \circ s_x$ are isometries fixing x which agree on T_xM so $s_x \circ t = t \circ s_x$ on M (since M is connected). We have therefore since G^t is closed, that G^t is transitive on M^t and hence $M^t = G^t/K^t$ where K^t is the typical isotropy subgroup at 0 say. If $\mathbf{g} = \mathbf{k} + \mathbf{m}$ is the Cartan decomposition with respect to $\mathrm{ad}(s_0)$, then since $\mathrm{ad}(s_0)$ and $\mathrm{ad}(t)$ commute by above and we see that $\mathrm{ad}(s_0)\circ\mathrm{ad}(t)(X) = \mathrm{ad}(t)\circ\mathrm{ad}(s_0)(X) = \pm\,\mathrm{ad}(t)(X)$. Thus $\mathrm{ad}(t)$ stabilizes the plus and minus one eigenspaces $\mathbf{k}$ and $\mathbf{m}$ of $\mathrm{ad}(s_0)$. We can therefore decompose $\mathbf{m}$ and $\mathbf{k}$ into their eigenspaces with respect to the involution $\mathrm{ad}(t)$, so that we have $\mathbf{m} = \mathbf{m}^t + \mathbf{m}^{-t}$ and $\mathbf{k} = \mathbf{k}^t + \mathbf{k}^{-t}$ (the sign convention being the obvious one). Letting $\mathbf{g}^t$ denote the Lie algebra of G^t, we have that $\mathbf{g}^t = \mathbf{k}^t + \mathbf{m}^t$ and we can identify $\mathbf{m}^t$ with the tangent space T_0M^t to M^t at 0. We can also identify $\mathbf{m}^{-t}$ with the tangent space T_0M^{-t} to M^{-t} at 0, where $M^{-t} = F(s_0 \circ t, M)(0)$, also a symmetric space.

Lemma 1.1. *We have the following orthogonal decompositions.*

$$\mathbf{m}^t = \sum_{\alpha \in R_t} \mathbf{m}^t_\alpha, \qquad m^{-t} = \sum_{\alpha \in R_{-t}} \mathbf{m}^{-t}_\alpha$$

where R_t and R_{-t} are sets of linear forms on $\mathbf{h}^{-t}$ a maximal abelian subalgebra of $\mathbf{m}^{-t}$, which are not necessarily distinct and may include the zero form.

Proof. Let $H \in \mathbf{h}^{-t} \subset \mathbf{m}^{-t}$, then $\mathrm{ad}(H)\mathbf{m}^{-t} = [H, \mathbf{m}^{-t}] \subset [\mathbf{m}, \mathbf{m}] \subset \mathbf{k}$ and $\mathrm{ad}(t)[H, \mathbf{m}^{-t}] = [\mathrm{ad}(t)H, \mathrm{ad}(t)\mathbf{m}^{-t}] = [-H, -\mathbf{m}^{-t}] = [H, \mathbf{m}^{-t}]$. Therefore by definition of $\mathbf{k}^t$ we have that $\mathrm{ad}(H)\mathbf{m}^{-t} \subset \mathbf{k}^t$. A similar argument shows that $\mathrm{ad}(H)\mathbf{k}^t \subset \mathbf{m}^{-t}$ and therefore we have that the commutative system of semisimple operators $\{\mathrm{ad}(H)^2 | H \in \mathbf{h}^{-t}\}$ stabilizes $\mathbf{m}^t$ and $\mathbf{m}^{-t}$. Note we also get the following decompositions of $\mathbf{k}^t$ and $\mathbf{k}^{-t}$.

$$\mathbf{k}^t = \sum_{\alpha \in R_{-t}} \mathbf{k}^t_\alpha, \qquad \mathbf{k}^{-t} = \sum_{\alpha \in R_t} \mathbf{k}^{-t}_\alpha.$$

This follows from the fact that $\mathrm{ad}(H)\colon \mathbf{m}^t_\alpha \to \mathbf{k}^{-t}_\alpha$ is a linear isomorphism if $\alpha \in R_{-t}$ and $\alpha(H) \neq 0$ and similarly for $\mathrm{ad}(H)\colon \mathbf{m}^{-t}_\alpha \to \mathbf{k}^t_\alpha$ with $\alpha \in R_t$. $\square$

Before proceeding we recall a few more facts. For $X \in \mathbf{g}$ we may consider X as a vector field on M, which we will also denote by X. It's value $X(p)$ at $p \in M$ is the initial tangent to the curve $(\exp tX)(p)$. Since G acts on M as a group of isometries, it therefore carries geodesics to geodesics and hence X restricted to a geodesic is a Jacobi field along that geodesic.

Given a compact Lie transformation group L acting on a manifold M, on which we have an L invariant metric, the orbit of L through x will be denoted by $L(x)$. The orbit $L(x)$ is a compact submanifold homeomorphic with L/L_x where L_x is the isotropy subgroup at x.

Definition. An orbit whose dimension is less than that of a maximal dimensional orbit will be called a singular orbit.

Theorem 1.2. *The conjugate locus of M^t in M is the union of the singular G^t orbits, that is $JL(M^t, M) = \{x \in M | G^t(x) \text{ is singular}\}$.*

Proof. Suppose $x \in JL(M^t, M)$, we may assume that x lies in a maximal torus in M^{-t} and that x is a conjugate point of M^t along a geodesic $\gamma = \gamma_H: s \to \operatorname{Exp} sH$ issuing from 0 with $H \in \mathbf{h}^{-t}$. We wish to consider Jacobi fields V along γ for which $V(\gamma(0)) \in T_0M^t$ and $V'(\gamma(0)) - S_{\gamma'}V(\gamma(0))$ is normal to M^t and therefore is contained in T_0M^{-t}. Since M^t is totally geodesic, the second fundamental form of M^t is zero and hence the second initial condition reduces to the condition $V'(\gamma(0)) \in T_0M^{-t}$. The set of such Jacobi fields which vanish for some parameter value $s_c > 0$ will be denoted by $\mathcal{J}^\gamma$, that is $\mathcal{J}^\gamma$ is the set of all Jacobi fields satisfying the following conditions.

$$V(\gamma(0)) \in T_0M^t, \qquad V'(\gamma(0)) \in T_0M^{-t}, \quad \text{and} \quad V(s_c) = 0.$$

We will show that all Jacobi fields in $\mathcal{J}^\gamma$ are the restrictions to γ of vector fields in $\mathbf{g}^t$. We define $V_0: \mathbb{R} \to T_0M$ by $V_0(s) =$ the parallel transport to 0 along γ of $V(\gamma(s))$. Since the curvature tensor is parallel we may identify V with V_0 and consider the Jacobi equation as an equation in T_0M. Since $R(X_\alpha, H)H = \alpha(H)^2X_\alpha$ at 0, we see that an element V of $\mathcal{J}^\gamma$ can be considered as a solution of the system of equations,

$$(V_\alpha^\pm)'' + \alpha(H)^2V_\alpha^\pm = 0 \quad \text{where } V = \sum_{\alpha \in R_t} V_\alpha^+ + \sum_{\alpha \in R_{-t}} V_\alpha^-$$

is the decomposition of V with respect to the decompositions of $\mathbf{m}^t$ and $\mathbf{m}^{-t}$. In addition the initial conditions $(V_\alpha^+)'(0) = 0$ and $V_\alpha^-(0) = 0$ must be satisfied. We consider three cases.

Case 1. $\alpha(H) = 0$. Then $V_\alpha^\pm(s) = V_\alpha^\pm(0) + s(V_\alpha^\pm)'(0)$, but since $(V_\alpha^+)'(0) = 0$ we have that $V_\alpha^+(s) = V_\alpha^+(0)$. However since $V_\alpha^+(s_c) = 0$ for some $s_c > 0$ we have that V_α^+ must be identically zero. $V_\alpha^-(0) = 0$ implies that $V_\alpha^-(s) = s(V_\alpha^-)'(0)$, therefore $V_\alpha^-(s_c) = 0$ implies that $(V_\alpha^-)'(0) = 0$ and we have that V_α^- is identically zero being a solution of a second order linear differential equation with both initial conditions equal to zero.

Case 2. $\alpha(H) \neq 0$ and $\alpha \in R_t$. In this case we choose $X_\alpha \in m_\alpha^t$ such that $V_\alpha^+(0) = X_\alpha$. Now $(X_\alpha)'_{|\gamma}(0) = [H, X_\alpha] + \nabla_{X_\alpha}H(0) = [H, X_\alpha]$ since $\nabla X = 0$ at 0 for $X \in \mathbf{m}$. Since $[H, X_\alpha] \in \mathbf{k}$ we have that $(X_\alpha)'_{|\gamma}(0) = 0$, as the vector fields from $\mathbf{k}$ all vanish at 0. We therefore have that $X_{\alpha|\gamma} = V_\alpha^+$ since they satisfy the same initial conditions.

Case 3. $\alpha(H) \neq 0$ and $\alpha \in R_{-t}$. Since $(V_\alpha)'(0) \in \mathbf{m}_\alpha^{-t}$ and $\alpha(H) \neq 0$ there exists a unique vector $Y_\alpha \in \mathbf{k}_\alpha^t$ such that $(Y_\alpha)'(0) = [H, Y_\alpha] = (V_\alpha^-)'(0)$ (the first equality holds because $Y_\alpha(0) = 0$). Also $V_\alpha^-(0) = 0$ implies that $Y_{\alpha|\gamma} = V_\alpha^-$ since they have the same initial conditions. We have shown therefore that every element of $\mathcal{J}^\gamma$ comes from $\mathbf{g}^t$, and thus if x is a conjugate point of M^t the dimension of the Lie algebra of the isotropy subgroup at x is greater than $\dim(k_0^t+m_0^t)$. From the formula $\dim G^t(x) = \dim G^t - \dim L(G_x^t)$ where $L(G_x^t)$ denotes the Lie algebra of the isotropy subgroup at x, and the fact that an orbit of maximal dimension has dimension given by $\dim G^t(x) = \dim G^t - \dim(\mathbf{k}_0^t + \mathbf{m}_0^t)$, we see that $G^t(x)$ is singular.

Conversely if $G^t(x)$ is singular then $\dim G^t(x) < \dim G^t - \dim(\mathbf{k}_0^t+\mathbf{m}_0^t)$ and therefore for some $\alpha \neq 0$, $\alpha \in R_t \cup R_{-t}$ there is a $Y_\alpha \in \mathbf{m}_\alpha^t + \mathbf{k}_\alpha^t$ such that $Y_\alpha(x) = 0$ and $Y_\alpha \neq 0$. Since there is a dense set of directions in $\mathbf{h}^{-t}$ from 0 to x, there exists some $H \in \mathbf{h}^{-t}$ such that $\alpha(H) \neq 0$, $\operatorname{Exp} H = x$, $Y_{\alpha|\gamma}$ is not identically zero along $\gamma = \operatorname{Exp} tH$, and $Y_\alpha(x) = 0$. This shows that x is conjugate to M^t along a geodesic γ and thus is contained in the conjugate locus of M^t in M. □

We note that for the special case where $t = s_0$ the above theorem gives the variational completeness of Bott and Samelson [B-S, 1].

Corollary 1.3. *We have the following description of the set* $\mathcal{J}^\gamma$, $\gamma = \operatorname{Exp} tH$,

$$\mathcal{J}^\gamma = \left\{ \left(\sum_{\substack{\alpha\in R_t \\ \alpha(H)s_c \in \pi/2(\mathbb{Z}-2\mathbb{Z})}} X_\alpha + \sum_{\substack{\alpha\in R_{-t} \\ \alpha(H)s_c\in\pi(\mathbb{Z}-0)}} Y_\alpha \right)_{|\gamma} : X_\alpha \in \mathbf{m}_\alpha^t,\ Y_\alpha \in \mathbf{k}_\alpha^t,\ s_c > 0 \right\}.$$

Proof. This follows immediately from the explicit descriptions of the Jacobi fields $X_{|\gamma}$ and which are given as follows.

$$X_{|\gamma} = \cos(\alpha(H)s)X, \qquad X \in \mathbf{m}_\alpha^t,\ \alpha \in R_t,\ \alpha(H) \neq 0,$$
$$Y_{|\gamma} = \sin(\alpha(H)s)[H, Y], \qquad Y \in \mathbf{k}_\alpha^t,\ \alpha \in R_{-t},\ \alpha(H) \neq 0.$$

Corollary 1.4. *The conjugate locus of* 0 *in* M *is the union of the singular* K *orbits, that is* $JL(\{0\}, M) = \{x \in M | K(x)$ *is singular* $\}$.

Proof. Take the involution t to be s_0, then $M^t = F(s_0, M)(0) = \{0\}$, $G^t = F(\operatorname{ad}(s_0), G)_{(1)} = K_{(1)}$ and now apply Theorem 1.2. □

Proposition 1.5.

$$\dim K(x) = \sum_{\alpha(H) \notin \pi\mathbb{Z}} \dim \mathbf{m}_\alpha = \sum_{\alpha(H) \notin \pi\mathbb{Z}} \dim \mathbf{k}_\alpha$$

where $H \in \mathbf{h}$, $e^H(0) = x$, *and* $\alpha \in R(M)$.

Proof. $Y(e^H(0)) = 0$ if and only if $e^H(e^{-H}Ye^H)(0) = 0$. Since e^H is a diffeomorphism, this is equivalent to the saying that $e^{-H}Ye^H(0) = 0$, that is $\operatorname{ad}(e^{-H})Y \in \mathbf{k}$. Now writing $Y = Y_0 + \sum Y_\alpha$ where $Y_0 \in \mathbf{k}_0$, $Y_\alpha \in \mathbf{k}_\alpha$, and $\alpha \in R(M)$, we see by Theorem A that Y vanishes at x if and only if $\operatorname{ad}(e^{-H})Y = \sum \cos\alpha(H)Y_\alpha - \sin\alpha(H)X_\alpha$ is a member of $\mathbf{k}$, that is $\alpha(H) \in \pi\mathbb{Z}$ for $X_\alpha, Y_\alpha \neq 0$. □

Recalling the definition of the fundamental Weyl simplex S for an irreducible M, we see that its boundary $\operatorname{bd}(S)$ is contained in the union of the hyperplanes $\alpha^{-1}(0)$, $\alpha \in \Sigma$, and $(\tilde{\alpha})^{-1}(\pi)$. Writing S as a disjoint union $S = \bigcup_{i\in I} S^i$ of it's open cells S^i, we see that vectors in the top dimensional cell do not sit on any of the above hyperplanes, vectors in a given $r-1$ dimensional cell sit on one hyperplane, vectors in a given $r-2$ dimensional cell sit on the same two hyperplanes and so forth. Since $M = K\,\mathrm{Exp}\,S$ and since the dimension of an orbit $K(x)$ depends only on the hyperplanes on which H sits, where $\mathrm{Exp}\,H = x$ (by Proposition 1.5), we see that the dimensions of the orbits $K(x)$ are the same for all $x \in M^i = K\,\mathrm{Exp}\,S^i$ and are therefore all locally diffeomorphic. Recall the following known result see [BR, 1].

Proposition 1.6. *Let K be a compact Lie transformation group acting on a manifold M, then for all $x \in M$ there exists a K invariant neighbourhood U of $K(x)$ such that U is a disc bundle over $K(x)$, and U is associated to the principal bundle $K_x \to K \to K(x)$.*

Note from Corollary 1.4 and Proposition 1.5 we have that $JL(\{0\}, M) = K\,\mathrm{Exp}\,\mathrm{bd}(S)$ where $\mathrm{bd}(S)$ is the boundary of the simplex S.

Using Proposition 1.6 and the discussion prior to it, we get the following sketch proof of a result due to Takeuchi [T, 1] and also Sakai [S, 1].

Proposition 1.7.

$$M = \bigcup_{i\in I} M^i \quad \textit{and} \quad JL(\{0\}, M) = \bigcup_{i\in I - i_0} M^i$$

where each M^i is a connected submanifold and i_0 is the index corresponding to the top dimensional cell of S. $\dim M^i \le \dim^i M - 2$ for all $i \in I - i_0$ and each M^i is diffeomorphic with a vector bundle over a compact manifold.

We now apply Theorem 1.2 to obtain descriptions of the conjugate loci of two very important classes of totally geodesic submanifolds. We will first give some basic facts about these submanifolds, for an indepth study and classification see [N-C, 1]. For each smoothly closed geodesic c through 0, we consider the antipodal point p of 0 on c. Denote by $M^+(p)$ the orbit $K_{(1)}(p)$, then $M^+(p) = F(s_0, M)(p)$ is a symmetric space. Note that s_0 fixes the point p and therefore $s_0 \circ s_p = s_p \circ s_0$, so that $\mathrm{ad}(s_p)$ stabilizes $\mathbf{k}$ and $\mathbf{m}$ giving us a Cartan decomposition $\mathbf{g} = (\mathbf{k}^+ + \mathbf{m}^+) + (\mathbf{k}^- + \mathbf{m}^-)$ at p. $M^+(p) = K/K^+$ and the tangent space to $M^+(p)$ at p can be identified with $\mathbf{k}^-$, that is $T_pM^+(p) = \{Y(p) | Y \in \mathbf{k}^-\}$. The normal space to $T_pM^+(p)$ is the tangent space to another connected totally geodesic submanifold denoted by $M^-(p)$. $M^-(p) = F(s_p \circ s_0, M)(p) = F(s_p \circ s_0, M)(0)$. $T_0M^-(p)$ can be identified with $\mathbf{m}^-$ and similarly at p. $M^-(p) = G^-/K_p$ where G^- is the connected subgroup of G given by the Lie subalgebra $\mathbf{k}^+ + \mathbf{m}^-$ and K_p is the isotropy subgroup of p in G^-. One very useful property of $M^-(p)$ is that it has the same rank as M. Applying Theorem 1.2 using p as origin instead of 0, $t = s_0$, $G^t = F(\mathrm{ad}(s_0), G)_{(1)} = K_{(1)}$ and $t = s_p \circ s_0$, $Gt = F(\mathrm{ad}(s_0 \circ s_p), G)_{(1)} = G^-$ we get the following propositions.

Proposition 1.8. $JL(M^+(p), M) = \{x \in M | K(x) \textit{ is singular}\}$.

Corollary 1.9. *$JL(M^+(p), M) = JL(\{0\}, M)$ and therefore we obtain the stratification of Proposition* 1.7 *for the conjugate locus of $M^+(p)$ in M.*

Proposition 1.10. $JL(M^-(p), M) = \{x \in M | G^-(x) \textit{ is singular}\}$.

2

In this section we will consider the first conjugate locus of a polar set $M^+(p)$, where M is assumed to be irreducible.

Lemma 2.1. *Let $H_i \in \mathbf{h}$ be defined by $\alpha_k(H_i) = \pi\delta_{ik}$ for all $\alpha_k \in \Sigma$, and let $\tilde{\alpha} = \Sigma n_i\alpha_i$ be the expression of the highest root of M in terms of the simple roots, then every polar set $M^+(p)$ can be described in one of the following ways:*
1. $M^+(p) = K\,\mathrm{Exp}\,1/2H_i$ *with* $n_i = 1$ *or* 2 *and* $p = \mathrm{Exp}\,1/2H_i$.
2. $M^+(p) = K\,\mathrm{Exp}\,1/2(H_t+H_s)$ *with* $n_t = n_s = 1$ *and* $p = \mathrm{Exp}\,1/2(H_t+H_s)$.

Proof. Since $M = K\,\mathrm{Exp}\,S$, a given $M^+(p)$ can be described as the K orbit of the image of a vector H in S under the exponential map $\mathrm{Exp}\colon T_0M \to M$. By definition of the polar sets, this point p must be antipodal to 0 on a circle (that is a smoothly closed geodesic) c. We therefore have that $\mathrm{Exp}(2H) \in K$. This implies that $\mathrm{ad}(\exp 2H)\mathbf{k} = \mathbf{k}$ and $\mathrm{ad}(\exp 2H)\mathbf{m} = \mathbf{m}$, which since $\mathrm{ad}(\exp 2H)X_\alpha = \cos\alpha(2H)X_\alpha - \sin\alpha(2H)Y_\alpha$ for $X_\alpha \in m_\alpha$ implies that $\alpha(2H) \in \pi\mathbb{Z}$ for all $\alpha \in R(M)$. Since $H \in S$ we need only consider two cases.

Case I. $0 < \tilde{\alpha}(H) < \pi$, this means that $0 < \tilde{\alpha}(2H) < 2\pi$ and therefore $\tilde{\alpha}(2H) = \pi$ and $\tilde{\alpha}(H) = \pi/2$. From this it follows that the only possible choices for H are those vectors $1/2H_i$ such that $\alpha_k(H_i) = \pi\delta_{ik}$ and $n_i = 1$.

Case II. $\tilde{\alpha}(H) = \pi$, in this case the only possibilities for H are $H = 1/2H_i$ with $n_i = 2$ or $H = 1/2(H_t + H_s)$ with $n_t = n_s = 1$. □

Recall now the definition of the bottom space of M denoted by M^*. The adjoint group of G is defined by $\mathrm{ad}\,G = G/C$ where C is the centralizer of $G_{(1)}$, M^* is then defined to be $\mathrm{ad}\,G/F(\mathrm{ad}(s_0), \mathrm{ad}\,G) = \mathrm{ad}\,G/K^*$. M^* is characterized by the property that every space locally isometric to M is a covering Riemannian manifold of M^*. In particular there is a locally isometric projection $\pi\colon M \to M^*$. We have also that K^* is its own normalizer in $\mathrm{ad}\,G$, and therefore the following three conditions are equivalent on M^*.

(1) $\mathrm{ad}(b)\mathbf{m} = \mathbf{m}$, (2) $\mathrm{ad}(b)\mathbf{k} = \mathbf{k}$, (3) $b \in K^*$.

In particular if $\mathrm{ad}(\exp 2H)\mathbf{k} = \mathbf{k}$ on M^* then $(\exp H)(0)$ is antipodal to 0. Therefore, we have that the converse of the last result is true on M^*, and we have the following proposition.

Proposition 2.2. *If $M = M^*$, then $K\,\mathrm{Exp}\,1/2H_i$ is a polar set if $n_i = 1$ or 2, or else it is the origin 0. If $n_t = n_s = 1$ then $K\,\mathrm{Exp}\,1/2(H_t + H_s)$ is a polar set or else it is the origin 0.*

Recall when we take $t = s_0$ and origin at p, $M^t = M^+(p)$ and $M^{-t} = M^-(p)$ and we denote $R_{\pm t}$ by $R_\pm$. The fact that $M^-(p)$ has the same rank as M means we may assume that the maximal torus of M is contained in $M^-(p)$, and therefore the decompositions of Lemma 1.1 imply that

$$\mathbf{m}^- = \mathbf{h} + \sum_{\alpha\in R_-} \mathbf{m}_\alpha^-, \qquad \mathbf{m}^+ = \sum_{\alpha\in R_+} m_\alpha^+$$

where R_+ and R_- are subsets of the roots of M and $R(M) = R_+ \cup R_-$. As yet it is not clear that $R_+ \cap R_- = \varnothing$, as part of a root space may lie in $\mathbf{m}^+$ and the rest of it in $\mathbf{m}^-$. We now show that this cannot occur.

Proposition 2.3. *$\alpha \in R_+$ if and only if $\alpha(H_0) \in \pi/2(\mathbb{Z} - 2\mathbb{Z})$ and $\alpha \in R_-$ if and only if $\alpha(H_0) \in \pi\mathbb{Z}$, where $M^+(p) = k \operatorname{Exp} H_0$ and $H_0 \in S$.*

Proof. $\mathbf{k}^+ = \mathbf{k}_0 + \sum_{\alpha \in R_-} \mathbf{k}_\alpha^+$ and $M^+(p) = K/K^+$ so $Y_\alpha \in \mathbf{k}^+$ if and only if $Y_\alpha(p) = 0$ where $p = \operatorname{Exp} H_0$, that is $Y_\alpha((\exp H_0)(0)) = 0$. This is equivalent to the condition that $\exp H_0(\exp -H_0 Y_\alpha \exp H_0)(0) = 0$, which since $\exp H_0$ is a diffeomorphism is equivalent to the condition that $(\exp -H_0 Y_\alpha \exp H_0)(0) = 0$, that is $\operatorname{ad}(\exp -H_0) Y_\alpha \in \mathbf{k}$. By Theorem A we have the following equation, $\operatorname{ad}(\exp -H_0) Y_\alpha = \cos \alpha(H_0) Y_\alpha - \sin \alpha(H_0) X_\alpha$ and therefore the condition that $\alpha \in R_-$ reduces to the condition that $\alpha(H_0) \in \pi\mathbb{Z}$. Since $\exp -H_0$ carries $M^+(p)$ to $F(s_p, M)(0)$ and since the tangent space to $M^+(p)$ is given by $\{Y_\alpha | Y_\alpha \in \mathbf{k}_\alpha^-, \alpha \in R_+\}$, and the tangent space to $F(s_p, M)(0)$ is given by $\mathbf{m}^+$, we see that $\alpha \in R_+$ if and only if $\operatorname{ad}(\exp -H_0) Y_\alpha \in \mathbf{m}_\alpha^+$. Again by Theorem A we have that $\operatorname{ad}(\exp -H_0) Y_\alpha = \cos \alpha(H_0) Y_\alpha - \sin \alpha(H_0) X_\alpha$ and therefore the condition that $\alpha \in R_+$ is equivalent to the condition that $\alpha(H_0) \in \pi/2(\mathbb{Z} - 2\mathbb{Z})$. From these two conditions it is now apparent that $R_+ \cap R_- = \varnothing$.

We now give a description of the first conjugate locus of an $M^+(p)$ on the bottom space in the case where the linear isotropy representation of G^- is absolutely irreducible on the tangent space of G/G^-. A description can be given in the other cases using the same type of analysis but it needs a case by case argument. We will also assume for simplicity that the root system of M is not A_r or that of E_6 and therefore the $M^+(p)$ comes from a $1/2H_i$ with $n_i = 1$ or 2.

Theorem 2.4. *Under the above assumptions the first conjugate locus of a polar set $M^+(p)$ is given as follows.*

(1) *$FL(M^+(p), M) = K \operatorname{Exp}_p\{(\tilde{\alpha})^{-1}(\pi/2) \cap C_p\}$ if $p = \operatorname{Exp} 1/2H_i$ and $n_i = 1$.*

(2) *$FL(M^+(p), M) = K \operatorname{Exp}_p\{(\tilde{\beta})^{-1}(\pi/2) \cap C_p\}$ if $p = \operatorname{Exp} 1/2H_i$ with $n_i = 2$, where $\tilde{\alpha}$ is the highest root of M, $\tilde{\beta}$ is the highest root in R_+, C_p is a closed positive Weyl chamber at p and $\operatorname{Exp}_p \colon T_pM \to M$ is the exponential map at p.*

Proof. We apply Theorem 1.2 taking p as our origin and $t = s_0$. We then have $M^t = M^+(p)$, $G^t = K_{(1)}$, and $g^t = k^+ + k^-$. Corollary 1.3 tells us that

$$\mathscr{J}_H^\gamma = \left\{ \left(\sum_{\substack{\alpha \in R_+ \\ \alpha(H)s_c \in \pi/2(\mathbb{Z} - 2\mathbb{Z})}} Y_\alpha + \sum_{\substack{\alpha \in R_- \\ \alpha(H)s_c \in \pi\mathbb{Z} - 0}} Z_\alpha \right)_{|\gamma} : Y_\alpha \in \mathbf{k}_\alpha^-,\ Z_\alpha \in \mathbf{k}_\alpha^+ \text{ and } s_c > 0 \right\}$$

describes all Jacobi fields along $\gamma = \gamma_H$ where γ is a geodesic issuing from $p = \operatorname{Exp} H_0$ with initial tangent H lying in a maximal abelian subalgebra $\mathbf{h}$ of m^-. By Proposition 2.3 $\gamma(s_f)$ is a first conjugate point of $M^+(p)$ along γ if and only if one of the following conditions is satisfied.

For all α such that $\alpha(H_0) \in \pi/2(\mathbb{Z} - 2\mathbb{Z})$, $\alpha(Hs) < \pi/2$ if $s \in (0, s_f)$ and for all α such that $\alpha(H_0) \in \pi\mathbb{Z}$, $\alpha(Hs) < \pi$ if $s \in (0, s_f)$, and either there exists an α with $\alpha(H_0) \in \pi/2(\mathbb{Z} - 2\mathbb{Z})$ such that $\alpha(Hs_f) = \pi/2$, or there exists an α with $\alpha(H_0) \in \pi\mathbb{Z}$ such that $\alpha(Hs_f) = \pi$. Restricting our attention to a positive Weyl chamber C_p we have that $\alpha(Hs_f) \leq \beta(Hs_f)$ if $\alpha < \beta$, α and β positive. We now consider the three possible descriptions for $M^+(p)$.

Case I. $H_0 = 1/2H_i$ with $n_i = 1$ and $M^+(p) = K \operatorname{Exp} 1/2H_i$.

In this case $\widetilde{\alpha}(1/2H_i) = \sum n_i\alpha_i(1/2H_i) = \pi/2$ and therefore $\widetilde{\alpha} \in R_+$ and $\widetilde{\alpha}(Hs) \geq \alpha(Hs)$ for all $\alpha \in R(M)$ and for all $s \in (0, s_f)$. If $\widetilde{\alpha}(Hs) < \pi/2$, then $\alpha(Hs) \leq \widetilde{\alpha}(Hs) < \pi/2 < \pi$ and therefore the first tangent conjugate locus of $M^+(p)$ in C_p is given by $(\widetilde{\alpha})^{-1}(\pi/2) \cap C_p$.

Case II. $H_0 = 1/2H_i$ with $n_i = 2$.

In this case $\widetilde{\alpha} \in R_-$ since if $H_0 = 1/2H_i$ with $n_i = 2$, $\widetilde{\alpha}(1/2H_i) = \sum n_i\alpha_i(1/2H_i) = \pi$. We now show that if $\widetilde{\beta}$ is the highest root in R_+, then $2\widetilde{\beta} \geq \widetilde{\alpha}$ and therefore if $\widetilde{\alpha}(Hs_f) = \pi$, we have that $\widetilde{\beta}(Hs_f) \geq \pi/2$ so that the first tangent conjugate locus of $M^+(p)$ in C_p is given by $(\widetilde{\beta})^{-1}(\pi/2) \cap C_p$. In order to show that $2\widetilde{\beta} \geq \widetilde{\alpha}$ we consider the cases where M is of classical type and exceptional type separately. We assume first that M is classical and $R(M) \neq D_r$, then $\langle \alpha_i, \alpha_{i+1} \rangle < 0$ and therefore $\alpha_i + \alpha_{i+1} \in R(M)$, $\langle \alpha_{i-1}, \alpha_i + \alpha_{i+1} \rangle = \langle \alpha_{i-1}, \alpha_i \rangle + 0 < 0$ and therefore $\alpha_{i-1} + \alpha_i + \alpha_{i+1} \in R(M)$. Continuing inductively we get that $\alpha_1 + \cdots + \alpha_r \in R(M)$. If $R(M) = D_r$ we apply the above argument to get that $\alpha_1 + \cdots + \alpha_{r-1} \in R(M)$, then $\langle \alpha_1 + \cdots + \alpha_{r-2} + \alpha_{r-1}, \alpha_r \rangle = \langle \alpha_{r-2}, \alpha_r \rangle < 0$ and we again see that $\alpha_1 + \cdots + \alpha_r \in R(M)$. Applying Corollary 2.3 we see that $\alpha_1 + \cdots + \alpha_r \in R_+$, and since in the expression $\widetilde{\alpha} = \sum n_i\alpha_i$ all $n_i \leq 2$ when M is of classical type, but not of type BC, we have that $2\widetilde{\beta} \geq \widetilde{\alpha}$ as linear forms on C_p. We next note that the roots in R_+ are the restrictions to $\mathbf{h}$ of the weights of the linear isotropy representation of G^- on the tangent space $m^+ + k^-$ to the symmetric space G/G^-, and therefore $\widetilde{\beta}$ is the dominant weight of this representation. Therefore in the case where M is of exceptional type or BC, we merely read the G^- off the list of Nagano and Chen [N-C, 1] and since the dominant weights $\widetilde{\beta}$ have been computed by Cartan [C, 1] we see that $2\widetilde{\beta} \geq \widetilde{\alpha}$ and the theorem follows. □

Note the same argument gives the following proposition.

Proposition 2.5. (1) $FL(F(s_p, M)(0), M) = K$, $\operatorname{Exp}\{(\widetilde{\alpha})^{-1}(\pi/2) \cap C\}$ *if* $p = \operatorname{Exp} 1/2H_i$ *with* $n_i = 1$.

(2) $FL(F(s_p, M)(0), M) = K$, $\operatorname{Exp}\{(\widetilde{\beta})^{-1}(\pi/2) \cap C\}$ *if* $p = \operatorname{Exp} 1/2H_i$ *with* $n_i = 2$, *where* $\widetilde{\alpha}$ *is the highest root of* M, $\widetilde{\beta}$ *is the highest root in* R_+ *and* C *is a closed positive Weyl chamber at* 0.

Corollary 2.6 [S, 1], [T, 1]. $FL(\{0\}, M) = K$, $\operatorname{Exp}\{(\widetilde{\alpha})^{-1}(\pi) \cap C\}$ *where* C *is a closed positive Weyl chamber at* 0, *and* $\widetilde{\alpha}$ *is the highest root of* M.

Proof. We apply Theorem 1.2 with $t = s_0$ and $M^t = \{0\}$ and therefore by Corollary 1.3 $\mathscr{J}^\gamma = \{(\sum Y_\alpha)_{|\gamma}: Y_\alpha \in k_\alpha,\ \alpha \in R(M)$ and $\alpha(Hs_c) \in \pi\mathbb{Z} - 0\}$ describes all Jacobi fields along $\gamma = \gamma_H$ which vanish at 0 and for some parameter value $s_c > 0$, the result now follows. □

3

In this section we will discuss the first conjugate locus of an arbitrary $M^-(p)$, and the applications of our theory to computing homology. In our discussion of $FL(M^-(p), M)$ we will denote by M^+ the copy of $M^+(p)$ at 0, that is $M^+ = F(s_p, M)(0)$. Recall that $M^-(p) = M^- = G^-(0) = G^-(p)$ where p

is the antipodal point on the circle defining the pair $(M^+(p), M^-(p))$. We denote by $\mathbf{g}^+$ the Lie subalgebra $\mathbf{k}^+ + \mathbf{m}^+$, and by G^+ the corresponding connected Lie subgroup of G. Since $\mathbf{m}^+ \cong \mathbf{k}^-$ we have that $\mathbf{k} \cong \mathbf{k}^+ + \mathbf{m}^+$, and $\mathbf{m}^+ + \mathbf{m}^- = \mathbf{m} \cong \mathbf{k}^- + \mathbf{m}^-$. Since $[\mathbf{k}^+ + \mathbf{m}^+, \mathbf{k}^- + \mathbf{m}^-] \subset \mathbf{k}^- + \mathbf{m}^-$ we see that the adjoint representation of $\mathbf{k}^+ + \mathbf{m}^+$ on $\mathbf{k}^- + \mathbf{m}^-$ is isomorphic to the linear isotropy representation of $K \cong G^+$ on the tangent space to M at 0. Choosing a maximal abelian subalgebra $\mathbf{h}^+$ in $\mathbf{m}^+$ the decompositions of Lemma 1.1 at 0 with respect to $t = \operatorname{ad}(s_p)\operatorname{ad}(s_0)$ are:

$$\mathbf{m}^+ = \mathbf{h}^+ + \sum_{\alpha\in R} \mathbf{m}^+_\alpha, \qquad \mathbf{m}^- = \mathbf{m}^-_0 + \sum_{\lambda\in\Lambda} \mathbf{m}^-_\lambda$$

where $R = R(M^+)$ are the roots of the symmetric space M^+ and Λ is the set of the weights of the adjoint representation of $\mathbf{g}^+$ on $\mathbf{m}^- + \mathbf{k}^-$ restricted to $\mathbf{h}^+$. By the above discussion these weights are the same as those of the linear isotropy representation of K on T_0M since the representations are isomorphic, we note that the dominant weights of these representations have been calculated by Cartan [C, 1]. Again we restrict our attention to the case where this representation is absolutely irreducible.

Lemma 3.1. *Let $\tilde{\lambda}$ be the dominant weight in Λ, and let $\tilde{\alpha}$ be the highest root of M^+, then $2\tilde{\lambda} \geq \tilde{\alpha}$.*

Proof. Consider the following composition of maps, where the first map is the bracket product and the second is just the obvious projection.

$$\mathbf{m}^- + \mathbf{k}^- \otimes \mathbf{m}^- + \mathbf{k}^- \to \mathbf{k}^+ + \mathbf{m}^+ \to \mathbf{m}^+.$$

Since m^+ is a direct sum of nonzero root spaces with respect to a maximal abelian subalgebra $\mathbf{h}^-$ of $\mathbf{m}^-$ the first map has in its image vectors with arbitrary $\mathbf{m}^+$ component, and therefore the composite is surjective. Taking the root space decomposition of $\mathbf{m}^+$ with respect to $\mathbf{h}^+$, and the weight space decomposition of $\mathbf{m}^- + \mathbf{k}^-$ described above, representation theory then tells us that every root is a sum of two weights. However each weight $\lambda \in \Lambda$ is of the form $\lambda = \tilde{\lambda} - \sum m_i\alpha_i$ where the α_i's are positive roots of M^+ (see [H, 2, Chapter 5]). We now have $\tilde{\alpha} = \tilde{\lambda} - \sum m_j\alpha_j + \tilde{\lambda} - \sum n_i\alpha_i \leq 2\tilde{\lambda}$. □

Theorem 3.2. *The first conjugate locus of $M^-(p)$ in M has the following description, $FL(M^-(p), M) = G^- \operatorname{Exp}\{(\tilde{\lambda})^{-1}(\pi/2) \cap C\}$ where C is a positive Weyl chamber in h^+ at 0.*

Proof. Let $\gamma_H = \gamma$ be a geodesic normal to $M^-(p)$ at 0 with initial tangent H which we may assume to lie in a closed positive Weyl chamber C in $\mathbf{h}^+$. Corollary 1.3 then tells us that

$$\mathscr{J}^\gamma = \left\{ \left(\sum_{\substack{\lambda\in\Lambda \\ \lambda(Hs_c)\in\pi/2(\mathbb{Z}-2\mathbb{Z})}} X_\lambda + \sum_{\substack{\alpha\in R \\ \alpha(Hs_c)\in\pi(\mathbb{Z}-0)}} Y_\alpha \right)_{|\gamma} : X_\lambda \in m^-_\lambda,\ Y_\alpha \in k^+_\alpha,\ s_c > 0 \right\}$$

describes all Jacobi fields along γ satisfying the necessary initial conditions for $M^-(p)$ and vanishing for some parameter value $s_c > 0$. Thus $\gamma(s_f)$ being a first conjugate point of $M^-(p)$ along γ is equivalent to one of the following conditions being satisfied.

For all $\lambda \in \Lambda$ we have $\lambda(Hs) < \pi/2$ for $s \in (0, s_f)$, and for all $\alpha \in R\alpha(Hs) < \pi$ for $s \in (0, s_f)$, and either there exists a $\lambda \in \Lambda$ with $\lambda(Hs_f) = \pi/2$ or there exists an $\alpha \in R$ with $\alpha(Hs_f) = \pi$. Since $H \in C$ we may use the previous lemma to conclude that the first conjugate point comes from the Jacobi fields $X_{\tilde{\lambda}}$ and therefore the first tangent conjugate locus of $M^-(p)$ in M intersected with the positive Weyl chamber is given by $(\tilde{\lambda})^{-1}(\pi/2) \cap C$. □

Corollary 3.3. *If $M^+(p)$ has rank one then $FL(M^-(p), M)$ is a submanifold.*

Proof. Consider the hyperplane $(\tilde{\lambda})^{-1}(\pi/2) = \{H \in \mathbf{h}^+ | \tilde{\lambda}(H) = \pi/2\}$, since $\mathbf{h}^+$ is one dimensional $= \mathbb{R}H_0$ where H_0 may be taken such that $\tilde{\lambda}(H_0) = 1$. Therefore H lies on the above hyperplane if and only if $H = \pi/2H_0$, and Theorem 3.2 tells us that $FL(M^-(p), M) = G^-(x_0)$ where $x_0 = \operatorname{Exp} \pi/2H_0$ and is a submanifold. □

Definition. Let N be a closed connected submanifold of a complete connected Riemannian manifold M. Then $x \in M$ is said to be a cut point of N along γ if $\gamma(0) = x$, $\gamma(t_1) \in N$, $t_1 > 0$ and the following two conditions hold.

(1) For all $t \in (0, t_1]$, γ restricted to $[t, t_1]$ is a shortest geodesic from $\gamma(t)$ to N, and

(2) for all $t < 0$, γ restricted to $[t, t_1]$ is not a shortest geodesic from $\gamma(t)$ to N. The cut locus of N in M, denoted by $\mathrm{CL}(N, M)$ is then defined as follows $\mathrm{CL}(N, M) = \{x \in M | x$ is a cut point of N along some geodesic$\}$.

Proposition 3.4. *If the pair $(M^+(p), M^-(p))$ is such that $M^+(p)$ has rank one, then the first conjugate locus of $M^-(p)$ and the cut locus of $M^-(p)$ agree, that is, $FL(M^-(p), M) = \mathrm{CL}(M^-(p), M)$.*

Proof. Let x_0 be as in the previous corollary, that is $G^-(x_0) = FL(M^-(p), M)$, and $x_0 \in A^+$ a maximal torus in M^+ at 0. It can be checked that $x_0 \neq 0$. Since M^+ has rank one A^+ is just a circle which we will denote by γ. Suppose γ_1 is a shortest geodesic from x_0 to $M^-(p)$, and therefore γ_1 is normal to $M^-(p) = G^-(0)$ at y say. Applying an element $b \in G^-$ such that $b(y) = 0$ to γ_1 gives us a geodesic at 0 which is again normal to $M^-(p)$ and is therefore contained in M^+. We can therefore apply an element $l \in K^+_{(1)} \subset G^-$ to $b\gamma_1$ so that $lb\gamma_1 \subset \gamma$. Since $lb(x_0)$ is in the same G^- orbit as x_0 Proposition 1.10 tells us it is a conjugate point of $M^-(p)$. However since x_0 is the first conjugate point of $M^-(p)$ along γ we have that $lb(x_0) = x_0$ and hence the first conjugate locus of $M^-(p)$ is contained in the cut locus of $M^-(p)$ because γ_1 was a shortest geodesic to $M^-(p)$. Suppose now that q is a cut point of $M^-(p)$ along some geodesic, then we again bring it into the maximal torus γ. Now from the above argument the point that q goes to in γc does not come before x_0, and therefore it must actually be x_0 so the corollary is proved.

We now illustrate the preceding theory using the example of $CI(2) = G_2^0(\mathbb{R}^5)$ the Grassmann manifold of oriented 2 planes in $\mathbb{R}^5$. We will use the theory of Morse as applied by Bott [B, 1] to compute the homology of this space. The same analysis can be applied to other spaces with a rank one $M^+(p)$. These include most classical spaces and exceptional spaces such as F_4, FI, and EIV. Also some exceptional spaces have other fixed point sets of involutions which have normal spaces of rank one to which we can apply the same theory.

Example 1. $M = G_2^0(\mathbb{R}^5)$. The pairs $(M^+(p), M^-(p))$ have been calculated for all symmetric spaces by Nagano and Chen [N-C, 1], and we will use their descriptions. We will use $M^-(p) = G_2^0(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \cong S^2 \times S^2$ where the e_i's are the standard basis for $\mathbb{R}^5$ $G = SO(5)$, $K = SO(2) \times SO(3)$. We take $0 = e_1 \wedge e_2$ as our origin and $p = e_3 \wedge e_4$. A direct computation then gives the descriptions $M^+(p) = G_2^0(e_3 \wedge e_4 \wedge e_5) \cong S^2$, $G^- = SO(4) \times SO(1)$. We will use $\gamma(t) = e_3 \wedge (\cos te_4 + \sin te_5)$ as our torus at $p = \gamma(0)$. For $b \in G^- b\gamma(t) = be_3 \wedge (\cos tbe_4 + \sin tbe_5)$ and therefore $b\gamma(t) = \gamma(t)$ if and only if $be_3 \wedge \cos tbe_4 - e_3 \wedge \cos te_4 = -be_3 \wedge \sin te_5 + e_3 \wedge \sin te_5$. Since the right-hand side of this equation contains an e_5 term but the left-hand side does not we see that these two 2 planes are orthogonal and therefore equality holds if and only if both are zero. This is equivalent to the following conditions being satisfied: $\cos tb(e_3 \wedge e_4) = \cos te_3 \wedge e_4$, and $\sin t(be_3 \wedge e_5) = \sin t(e_3 \wedge e_5)$. We may assume that $\sin t \neq 0$ since in that case $\gamma(t) \in M^-(p)$. Now $b\gamma(t) = \gamma(t)$ implies that $be_3 = e_3$ and $\cos te_3 \wedge be_4 = \cos t(e_3 \wedge e_4)$ and therefore we get a larger isotropy subgroup giving us a singular G^- orbit if and only if $\cos t = 0$ (excluding $M^-(p)$ of course). By Theorem 1.2 and the fact that there is only one singular G^- orbit apart from $M^-(p)$ itself, we have that this singular orbit is the first conjugate locus of $M^-(p)$ and therefore by Proposition 3.4 it is also the cut locus of $M^-(p)$. Now if $\cos t = 0$ then $b \in SO(e_1 \wedge e_2 \wedge e_4) \cong SO(3)$ and $G^-(\gamma(t)) = SO(4)/SO(3) \cong S^3$. We therefore have that $\mathrm{CL}(M^-(p), M) \cong S^3$. Note that it is a manifold. We can now use (essentially) the square of the distance from $M^-(p)$ as a Morse function on M whose critical submanifolds are just $M^-(p)$ and the cut locus S^3. Since the null space of the hessian at critical points on S^3 and $M^-(p)$ are just the tangent spaces at these points to the appropriate submanifold, we have that the critical submanifolds are nondegenerate. We now have from the work of Bott [B, 1] that $M = M^-(p) \cup \xi_3(S^3)$ where $\xi_3(S^3)$ is a plane bundle over S^3. We now compute how to attach this cell to $M^-(p)$. To do so we consider the flow lines of the gradient of our Morse function that is the geodesics normal to $M^-(p)$. Take a cellular decomposition of $S^3 = e^0 \cup e^3$ where $e^0 = e_1 \wedge e_5$, the appropriate torus passing through e^0 is given by $e_1 \wedge (\cos te_2 + \sin te_5)$, and it intersects $M^-(p)$ when $\sin t = 0$. Applying the isotropy subgroup of $e_1 \wedge e_5$ to $e_1 \wedge (\cos 0e_3 + \sin 0e_5)$ we see that the intersection of the flow lines into $e_1 \wedge e_5$ with $M^-(p)$, that is $\mathrm{bd}(e^3)$, is given by $\{e_1 \wedge be_2 | b \in SO(1) \times SO(3) \times SO(1)\} \cong G_1^0(e_2 \wedge e_3 \wedge e_4) \cong S^2$. Note if $be_2 = -e_2$ we are at the point $e_1 \wedge -e_2$ which is the other M^+ of M, called the pole of $CI(2)$. This point is antipodal to 0 on the product of Helgason spheres which give $M^-(p)$ $(M^-(p) = S^2 \times S^2)$, and therefore $\mathrm{bd}(e^3) \cong S^2$ sits diagonally in the product of Helgason spheres. This means that e^3 bounds one of these spheres but not both as they are independent. The cell $e^3 \times e^3$ is trivially a cycle since its boundary has dimension five and $M^-(p)$ has dimension four. These observations now give the following description of the homology of $CI(2)$.

$$H_*(CI(2), \mathbb{Z}) \cong \mathbb{Z}[0] + \mathbb{Z}[S^2] + \mathbb{Z}[M^-(p)] + \mathbb{Z}[M],$$

that is

$$H_*(CI(2), \mathbb{Z}) \cong H_0(CI(2), \mathbb{Z}) + H_2(CI(2), \mathbb{Z}) + H_4(CI(2), \mathbb{Z}) + H_6(CI(2), \mathbb{Z}).$$

Example 2.

$$\mathrm{CL}(\mathrm{Spin}(9), F_4) = (\mathrm{Spin}(9) \times \mathrm{Spin}(9))/\mathrm{Spin}(8).$$

Since Spin(9) is an M^- of F_4 with corresponding $M^+ = FII$ which is a space of rank one we have by Proposition 3.4 that the cut locus is the same as the first conjugate locus and is therefore a G^- orbit of a point x_0 in the maximal torus of the copy of M^+ at the identity element of the group F_4. The G^- in this case is $\mathrm{Spin}(9) \times \mathrm{Spin}(9)$ and its elements (b, c) act on a point $x \in F_4$ via $x \to bxc^{-1}$. Let $x_0 = \gamma(t_0)$ be a point in the maximal torus of the above FII and let $\gamma(t)$ be a minimizing geodesic from x_0 to Spin(9) which intersects Spin(9) at the identity element at time $t = 0$. We can always find such an x_0 since Spin(9) is the G^- orbit of the identity element and any minimizing geodesic to Spin(9) will be normal to Spin(9), hence will still be normal when brought to the identity by an element of G^-. Since the FII is totally geodesic this geodesic will lie in the FII and will therefore intersect its maximal torus at some point x_0. We now note that if there exists an element $(b, c) \in G^-$ such that $bx_0c^{-1} = x_0$ with $b \neq c$ then $x_0 \in \mathrm{CL}(\mathrm{Spin}(9), F_4)$, as the element (b, c) will move $\gamma(0)$ which is the identity, but will fix x_0, giving us at least two minimizing geodesics to Spin(9) from x_0. We have therefore that the cut locus is the G^- orbit of such an x_0. We will now determine such x_0's. Let σ_{II} be the involution defining the space FII. Then $\mathrm{Spin}(9) = F(\sigma_{II}, F_4)$ and FII containing the identity element is given by $FII = F(\sigma_{II} \circ s_1, F_4)$. We see therefore that $X_0 \in FII$ if and only if $\sigma_{II}(x_0) = x_0^{-1}$, so that $bx_0c^{-1} = x_0$ if and only if $bx_0c^{-1} = x_0^{-1}$. If $y = bx_0b^{-1}$ then we see that if $x_0 \in FII$ is such that $bx_0c^{-1} = x_0$ then $x_0^2 = bx_0c^{-1}x_0 = bx_0b^{-1}bc^{-1}x_0 = ybc^{-1}x_0 = ybx_0^{-1}b^{-1}x_0 = y^2$. We have therefore that $x_0^2 = \gamma(2t_0) = b\gamma(2t_0)b^{-1} = y^2$. Since $x \neq y$ this can only happen if x^2 sits on the M^+ of the space FII which by the list of Nagano and Chen [N-C, 1] is an S^8 with corresponding M^- which is also an S^8. We have therefore that x_0 and y sit on the equator S^7 of the latter S^8. Conversely if y and x_0 sit on this S^7 there exists $b \in \mathrm{Spin}(8)$ such that $y = bx_0b^{-1}$. Since $\sigma_{II}(y^{-1}x_0) = yx_0^{-1}$ and since $y^2 = x_0^2$ we have that $\sigma_{II}(y^{-1}x_0) = y^{-1}x_0$ and therefore $y^{-1}x_0$ is an element of Spin(9). Thus we have the existence of a unique element $c \in \mathrm{Spin}(9)$ such that $y^{-1}x_0 = bc$, that is $x_0 = ybc = bx_0b^{-1}bc = bx_0c$ with $b \neq c$. We now have that the isotropy subgroup of such x_0's is Spin(8) thus giving the desired result.

As in the previous example we get a Morse function with only two critical submanifolds corresponding to the max and min. The actual attaching maps are more difficult to compute but the following inequalities for any coefficient field K are immediate from the Morse-Bott inequalities.

Proposition 3.5.

$$b_i(F_4; K) \leq b_i(\mathrm{Spin}(9); K) + b_{i-8}(\mathrm{Spin}(9) \times \mathrm{Spin}(9)/\mathrm{Spin}(8), K).$$

Example 3.

$$\mathrm{CL}(G_4^0(\mathbb{R}^9), FI) = \mathrm{CL}(\mathrm{Spin}(9), F_4) \cap FI.$$

Let $Q\colon FI \to F_4$ be the quadratic representation. Since FI has no poles Q is an embedding [N-C, 1] and we see from the list of Nagano and Chen [N-C], by noting that M^- corresponding to Spin(9) in FI, that $Q^{-1}(\text{Spin}(9)) = G_4^0(\mathbb{R}^9)$. Again we obtain the usual Morse inequalities.

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DUBLIN INSTITUTE OF TECHNOLOGY, KEVIN STREET, DUBLIN 8, IRELAND
Current address: Department of Mathematics, University College, Galway, Ireland
E-mail address: matburns@bodkin.ucg.ie